$$h = \overline{h} = \rho_5(\eta J_0 + J_1) + \rho_6 J_1 - (1/4)(3\eta J_0 - J_1)\Psi_r - (\eta/4)(J_0 - \eta J_1)\Psi,$$

$$\rho_5 \equiv \frac{1}{12}\left[2\Psi + \Psi_r + \frac{BF(b)\, I_1(k_0)}{k_0 I_1'(k_0 b)}\right], \quad \rho_6 \equiv -\frac{1}{2}\left[F(1) - \frac{BF(b)\, I_1'(k_0)}{2I_1'(k_0 b)}\right].$$

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CONVECTIVE MOTIONS OF A FLUID IN A NEARLY SPHERICAL CAVITY WHEN HEATED FROM BELOW

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1. It is known [1] that a nonuniformly heated fluid can be in mechanical equilibrium only if the temperature gradient in it is vertical and has a constant value. Such a situation can occur, for example, in a spherical cavity in a solid mass with a vertical (downward) temperature gradient specified at infinity.

Let us consider the effect of a nearly spherical cavity on convective stability. Suppose the equation of the surface of the cavity is $r = 1 + sP_2^{(1)}\cos\varphi$, where the $P_e^{(m)}(\vartheta)$ are associated Legendre polynomials, r, ϑ, φ are polar coordinates, the radius R_0 of the undeformed sphere is taken as unity, and $s \ll 1$. This special shape of the cavity was chosen since $P_2^{(1)}\cos\varphi$ is one of the large-scale spherical harmonics whose presence in the spectrum of functions specifying the shape of the actual cavity leads to distortion of the isotherms in the fluid, and consequently to convective motion for arbitrarily small temperature gradients.

We write the equations of steady-state convection in dimensionless form, choosing as units of velocity, pressure, and temperature $g\beta AR_0^3/\nu$, $\rho g\beta AR_0^2$, and AR_0 respectively, where ρ, β, and ν are respectively the density, coefficient of thermal expansion, and kinematic viscosity, $\mathbf{g} = g\mathbf{k}$ is the acceleration due to gravity, and $\mathbf{A} = A\mathbf{k}$ is the constant temperature gradient at infinity. Then the equations and boundary conditions for the dimensionless velocity $\mathbf{v}$, pressure p, and temperatures T_1 and T_2 in the solid mass in the fluid for steady-state motion take the form [1]

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$$\mathrm{Gr}(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \Delta\mathbf{v} - T_2\mathbf{k},\ \operatorname{div}\mathbf{v} = 0,$$
$$\mathrm{Gr}\,\mathrm{Pr}\,\mathbf{v}\nabla T_2 = \Delta T_2,\ \Delta T_1 = 0,$$
$$r \to \infty,\ \nabla T_1 = k, \tag{1.1}$$
$$r = 1 + sP_2^{(1)}\cos\varphi,\quad \mathbf{v} = 0,\quad T_1 = T_2,\quad \varkappa\frac{\partial T_2}{\partial n} = \frac{\partial T_1}{\partial n}.$$

Here Gr = $g\beta AR_0^4\nu^{-2}$ and Pr = ν/χ are the Grashof and Prandtl numbers. $\varkappa = \varkappa_2/\varkappa_1$ is the ratio of the thermal conductivities of the fluid and solid mass, and n is a unit vector normal to the surface $r = 1 + sP_2^{(1)}\cos\varphi$.

We seek the solution of the problem in power series in a small parameter ε:

$$\mathbf{v} = \varepsilon\mathbf{v}^{(1)} + \varepsilon^2\mathbf{v}^{(2)} + \varepsilon^3\mathbf{v}^{(3)} + \ldots,$$
$$T_2 = T_2^{(0)} + \varepsilon T_2^{(1)} + \varepsilon^2 T_2^{(2)} + \varepsilon^3 T_2^{(3)} + \ldots, \tag{1.2}$$
$$T_1 = T_1^{(0)} + \varepsilon T_1^{(1)} + \varepsilon^2 T_1^{(2)} + \varepsilon^3 T_1^{(3)} + \ldots,$$
$$s = \varepsilon s_1 + \varepsilon^3 s_3 + \ldots$$

The last of Eqs. (1.2) defines the parameter ε. We find the successive terms of the expansions by the Bubnov–Galerkin method [1].

2. The zeroth approximation is known [1]. We seek the first approximation in terms of vector spherical harmonics [2, 3]:

$$T_2^{(1)} = \left\{\frac{\mathrm{Ra}}{28}r^5 - \frac{\mathrm{Ra}}{10}r^3 + \left[\frac{18+17\varkappa}{(2+\varkappa)\,140}\mathrm{Ra} + \frac{27s_1(\varkappa-1)}{5(2+\varkappa)^2}\right]r\right\}P_1^{(1)}\cos\varphi,$$
$$T_1^{(1)} = \left\{\frac{2\varkappa\,\mathrm{Ra}}{35(2+\varkappa)} - \frac{9s_1}{5}\left(\frac{\varkappa-1}{2+\varkappa}\right)^2\right\}\frac{P_1^{(1)}\cos\varphi}{r^2} + \frac{6s_1}{5}\,\frac{1-\varkappa}{2+\varkappa}\,\frac{P_3^{(1)}\cos\varphi}{r^4}, \tag{2.1}$$
$$\mathbf{v}^{(1)} = (r^3 - r)\,\mathbf{r}\times\nabla\left(P_1^{(1)}\sin\varphi\right).$$

Here Ra = $3g\beta AR_0^4/(2+\varkappa)\nu\chi$ is the Rayleigh number, determined by the temperature gradient in the cavity $3A/(2+\varkappa)$. It is known [1] that the basic critical motion in a spherical cavity has a structure similar to $\mathbf{v}^{(1)}$. The system of functions (2.1) satisfies the boundary conditions and the equations of continuity and heat-conduction exactly. To determine the parameter s_1, we multiply the Navier–Stokes equation by $\mathbf{v}^{(1)}$, and integrate over the volume of the cavity. We express the result in terms of the critical Rayleigh number Ra_*, determined from the linear theory for a spherical cavity [1]:

$$\frac{27s_1(\varkappa-1)}{100(2+\varkappa)^2} = 1 - \frac{\mathrm{Ra}}{\mathrm{Ra}_*},\quad \mathrm{Ra}_* = \frac{17325(2+\varkappa)}{37+68\varkappa}. \tag{2.2}$$

3. The form of the functions of the second approximation $\mathbf{v}^{(2)}$, $T_2^{(2)}$, and $T_1^{(2)}$ is determined by the boundary conditions and the inhomogeneous terms in (1.1):

$$(\mathbf{v}^{(1)}\nabla)\,\mathbf{v}^{(1)} = -\frac{(r^3-r)^2}{6r}\left[\left(2P_2 + 4P_0 + P_2^{(2)}\cos 2\varphi\right)\mathbf{r}_1 + r\nabla P_2 + \frac{1}{2}r\nabla\left(P_2^{(2)}\cos 2\varphi\right)\right],\quad \mathbf{v}^{(1)}\nabla T_2^{(1)} \sim P_1. \tag{3.1}$$

In accordance with (3.1) we seek the solution of the second approximation in the form

$$\mathbf{v}^{(2)} = fP_2\mathbf{r}_1 + gr\nabla P_2 + FP_2^{(2)}\cos 2\varphi\,\mathbf{r}_1 + Gr\nabla\left(P_2^{(2)}\cos 2\varphi\right) + h\mathbf{r}\times\nabla\left(P_3^{(2)}\sin 2\varphi\right). \tag{3.2}$$

We choose the functions f, g, F, G, and h so that they satisfy the condition v = 0 on the surface $r = 1 + s_1P_2^{(1)}\cos\varphi\,\varepsilon$ and the equation of continuity, and contain one Galerkin function with an adjustable coefficient α:

$$f = \frac{\mathrm{Ra}\,\alpha}{84}(r^5 - 2r^3 + r) - 3s_1(r^3 - r),$$

$$F = \frac{\mathrm{Ra}\,\alpha}{168}(r^5 - 2r^3 + r) + \frac{s_1}{2}(r^3 - r),$$

$$g = \frac{1}{6r}(fr^2)^1, \quad G = \frac{1}{6r}(Fr^2)^1, \quad h = -\frac{s_1 r^3}{15}.$$

For the second approximation we substitute Eq. (3.2) into the heat-conduction equations and solve them exactly. We substitute the expression for $T_2^{(2)}$ found in this way into the Navier–Stokes equation, and after multiplying it by the Galerkin function $\mathbf{v}_2'^{(2)}$ and integrating over the volume of the cavity, we obtain

$$\alpha = \left[\frac{4}{39}\,\frac{2+\varkappa}{\mathrm{Pr}} + \mathrm{Ra}\,\frac{4(64+83\varkappa)}{1044225} + \frac{9}{10}\,\frac{s_1^2(\varkappa-1)}{\mathrm{Ra}\,(2+\varkappa)(4+3\varkappa)} + s_1\frac{1544+7922\varkappa+4709\varkappa^2}{40950\,(2+\varkappa)(4+3\varkappa)}\right]\Bigg/\left[1 - \mathrm{Ra}\,\frac{140+157\varkappa}{405405\,(4+3\varkappa)}\right].$$

4. In the third approximation we require only one harmonic to determine s_3:

$$\mathbf{v}^{(3)} = \left[\beta(r^3 - r) + \frac{207}{35}rs_1^2\right]\mathbf{r}\times\nabla P_1^{(1)}\sin\varphi + \frac{12\,\mathrm{Ra}\,\alpha s_1}{441}(r - r^3)P_2^{(1)}\cos\varphi\mathbf{r}_1 + \frac{2\mathrm{Ra}\,\alpha s_1}{441}(3r - 5r^2)r\nabla P_2^{(1)}\cos\varphi,$$

$$T_1^{(3)},\ T_2^{(3)} \sim P_1^{(1)}\cos\varphi. \tag{4.1}$$

Following the usual procedure [1] for determining $T_1^{(3)}$ and $T_2^{(3)}$, we multiply the Navier–Stokes equation by the Galerkin function $(r^3 - r)\mathbf{r}\times\nabla P_1^{(1)}\sin\varphi$ and integrate over the volume of the sphere to find s_3:

$$\begin{aligned}\frac{\varkappa-1}{2+\varkappa}s_3 &= 1.0963\cdot10^{-4}(2+\varkappa)^2\mathrm{Ra}^2\alpha\,\mathrm{Pr}^{-1} + \mathrm{Ra}^3\cdot10^{-6}(3.2529 + \\ &+ 12.943\varkappa + 19.112\varkappa^2 + 10.566\varkappa^3) + \mathrm{Ra}^3\alpha\cdot10^{-7}(5.2409 + 9.4173\varkappa + \\ &+ 3.3984\varkappa^2) + \frac{\mathrm{Ra}^2 s_1}{2+\varkappa}\cdot10^{-4}(2.9919 + 8.3894\varkappa + 10.653\varkappa^2 + 9.0451\varkappa^3) \\ &+ \frac{\mathrm{Ra}\,s_1^2}{(2+\varkappa)^2(4+3\varkappa)}(1.7093 + 4.7022\varkappa + 5.1393\varkappa^2 + 2.2202\varkappa^3 + 0.22908\varkappa^4) + \\ &+ \frac{\alpha\,\mathrm{Ra}^2 s_1\cdot10^{-4}}{(2+\varkappa)(4+3\varkappa)}(14.057 + 25.259\varkappa + 15.307\varkappa^2 + 3.0962\varkappa^3) - \frac{s_1^3(\varkappa-1)}{(2+\varkappa)^3(4+3\varkappa)}(59.131 + 36.828\varkappa + 10.377\varkappa^2 + 10.663\varkappa^3).\end{aligned} \tag{4.2}$$

It is clear from (2.2) and (4.2) that for $\varkappa = 1$ (equal thermal conductivities of the solid mass and fluid) motion in the cavity begins only for $\mathrm{Ra} = \mathrm{Ra}_*$, and increases with increasing Ra, where $\mathrm{Ra} = \mathrm{Ra}_* + \varepsilon^2\mathrm{Ra}_2$, $\mathrm{Ra}_2 = 0.27s_3(\varkappa - 1)(2 + \varkappa)^{-2}$

The question of the stability of the flow in this problem arises not only for $\varkappa = 1$, but also for $s = 0$ (perfect sphere). In this case $s_1 \equiv s_3 \equiv 0$, and our results agree with the known sums of the solution of the corresponding problems [1].

To compare with experiment [4], and to illustrate the theory graphically, we consider a nearly spherical cavity in Plexiglas filled with water ($\varkappa = 3.26$, $\mathrm{Pr} = 7$). The corresponding graphs of ε as a function of Ra are shown in Fig. 1 ($s = 0$, 0.01, and 0.1 for curves 1-3 respectively). As one should expect, the curves have a form characteristic for this kind of problem [5]. Since motions and their stability for negative values of ε for $\mathrm{Ra} > \mathrm{Ra}_*$ were

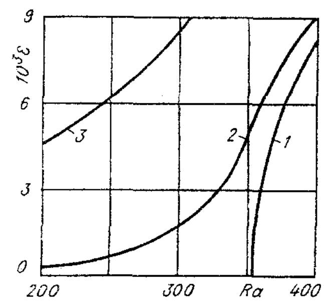

Fig. 1

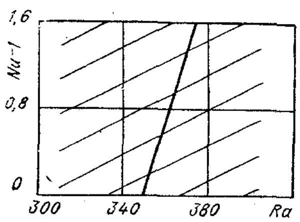

Fig. 2

studied in general form in [5], we do not consider these questions here. Figure 2 illustrates the dependence of the Nusselt number Nu (dimensionless convective heat flux through the cavity) on the Rayleigh number Ra for a perfect sphere $s \equiv 0$ (theory:solid line; the region containing the experimental points from [4] is shaded). Near Ra_* theory gives a linear dependence of $Nu - 1$ on $Ra - Ra_*$, which differs from the experimental results in [4].

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EXCITATION OF UNSTABLE WAVES IN BOUNDARY LAYER ON A VIBRATING SURFACE

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Modern concepts [1] on boundary-layer transition make it possible to develop a computational method for transition Reynolds numbers which includes the analysis of the growth of unstable disturbances in the boundary layer and the determination of the section at which their amplitude initially attains the critical value. Here, in order to develop a closed computational scheme it is necessary to solve the problem of excitation of the so-called Tollmien–Schlichting waves in the boundary layer. In experimental and theoretical studies [2-6], it has been shown that the Tollmien–Schlichting wave can arise due to flow nonuniformities of various types (sharp leading edge of the model, individual roughness element on the surface, localized effect on boundary layer). These results are discussed in sufficient detail in [7]. The adiabatic type excitation mechanism caused by natural, weak flow nonuniformity in the boundary layer on a smooth surface was suggested in [8]. A comprehensive qualitative and quantitative analysis of different types of excitation of Tollmien–Schlichting waves is necessary to solve applied problems. The present paper considers the excitation of unstable waves in boundary layer on a vibrating surface. The formulation of such a problem is discussed in [1].

Problem Formulation. Consider a two-dimensional incompressible boundary layer. Small differences arising from compressibility will be shown later. The coordinate system chosen is: x, distance from the leading edge of the model, downstream along the surface; y, distance normal to the surface; the reference scales are: certain length x_0 for the coordinate x, $\sqrt{\nu x_0/U_0}$ for y, where ν is the coefficient of kinematic viscosity, U_0 is the characteristic free stream velocity. Time is defined in units of $\sqrt{\nu x_0}/U_0^{3/2}$, pressure in terms of $\rho_0 U_0^2$, where ρ_0 is the density. Assume that the mean flow is weakly nonuniform in the absence of disturbances, i.e., for the streamwise and normal components of velocity U and V*, respectively, there are relations $U = U(x, y)$, $V^* = \varepsilon V(x, y)$, $\varepsilon = Re^{-1} = \sqrt{\nu/U_0 x_0} \ll 1$. Linearized Navier–Stokes equations after Fourier transformation in time are written in the form [8]

$$\frac{\partial \mathbf{A}}{\partial y} - H_1\mathbf{A} = \varepsilon H_2 \frac{\partial \mathbf{A}}{\partial x} + \varepsilon H_3 \mathbf{A}, \tag{1}$$

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